

GENERALIZED FUNCTIONS OF SLOW GROWTH
PROBLEM STATEMENT FOR CAPILLARY
WAVE FORMATION IN GAS—LIQUID
INTERFACE UNDER ULTRASONIC CAVITATION

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Abstract: The model of the formation of linear short capillary waves on the liquid-gas surface under the action of cavitation created by ultrasonic vibrations was proposed. The equations of propagation of capillary waves were constructed in the formulation of classical and generalized functions (of slow growth) that take into account: the viscosity of the liquid phase; attenuation of wave vibrations over time due to the viscosity of the liquid phase, which implies a limited amplitude of the waves (despite the fact that in the absence of attenuation, the wave can oscillate indefinitely over time). It was proved, that for equations in generalized functions for the case of collapse of a set of bubbles in a limited volume of liquid, the displacement profile (as generalized function, with is integral in the sense of the principal Cauchy value) of the interfacial surface is a regular generalized function of slow growth. The estimated dependences of the average increase in the interfacial surface on the parameters of ultrasonic action and the viscosity of the liquid are constructed. The dependences showed an increase in the interfacial surface up to 1.6 times or more for a liquid with a viscosity close to water. The obtained value is similar to the experimental data. The existence of a limiting viscosity has been established, starting from which the effect ceases to be noticeable. This indicates the need for research at different ambient temperatures. Since, on the one hand, with increasing temperature, the viscosity of the liquid phase decreases, and on the other hand, the degree of cavitation development decreases. Apparently, there may be an optimal temperature in this regard.

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1. Introduction

The mass transfer processes of a gas dissolved in a liquid are slow diffusion processes compared to the mass transfer of a gas dissolved in a gas or a liquid dissolved in another liquid. This is due to the low diffusion coefficient of the gas in the liquid, which is 5,000...10,000 times less than the gas diffusion coefficient in the gas [1, 2].

Due to this physical limitation, it is necessary to increase the total mass transfer rate by increasing the interfacial surface.

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The most obvious way to increase the interfacial surface gas-liquid is to spray the liquid into as small droplets as possible [3, 4]. On the one hand, spraying liquid into small droplets provides a huge interfacial surface compared to all other methods of forming an enlarged surface (without phase transition). On the other hand, spraying liquid to intensify mass transfer processes has a disadvantage, which consists in a low volume content of liquid in a gas-dispersed system. In this case, the liquid is very quickly saturated with the absorbed gas (despite the greatly increased mass transfer rate at the initial stage), and the process stops. Therefore, the basic physical principle of accelerating mass transfer, considered further, consists in the formation of stable capillary waves on a free surface gas-liquid with a volume of liquid significantly exceeding the volume of gas.

One of the most effective ways to implement periodic force action is the formation of ultrasonic vibrations in the cavitation mode [4–8].

Despite the fact that the first results indicating the effectiveness of ultrasonic exposure were obtained back in 1956 (ultrasonic intensification of benzene vapors with oil, performed at the Dnepropetrovsk Chemical Technology Institute [9]), there are currently no consistent studies of the physical mechanisms of the process.

Numerical and experimental results on the formation of capillary waves on the surface of a liquid up to high-amplitude waves capable of converting a liquid to a dispersed state are given in publications [10–22] (Monash University & Melbourne Centre for Nanofabrication, Australia; Imperial College, London, United Kingdom; Environmental Research Institute of Michigan & Naval Surface Warfare Center, USA; University of Notre Dame, USA; The Australian National University, Canberra, Australia; Columbia University, New York, USA; Petra Christian University, Indonesia; Klaipeda University, Lithuania; University of Amsterdam, The Netherlands).

In a publication by authors from Monash University & Melbourne Centre for Nanofabrication, Australia; Imperial College, London [10], a method for modeling the formation of nonlinear surface waves during excitation of liquid vibrations using a piezoelectric element was proposed. The method is based on decomposition into a number of physical parameters of a liquid by degrees of density, entropy and characteristic amplitude of the displacement velocity and solving the resulting system of equations of conservation of mass and momentum for the expansion coefficients by the finite difference method of the 2nd order. As a result of the simulation, wave profiles in the spatial domain and the frequency decomposition of the liquid surface were found. Experimental studies of the generated waves using a laser Doppler vibrometer have been carried out. The experimental data obtained are consistent with the theoretical values. Therefore, despite the fact that the presented publication does not consider diffusion processes, it is of methodological interest for the research from the point of view of modeling the formation of capillary waves when vibrations of a given surface are excited in non-resonant modes. The method described in the presented publication makes it possible to determine the resonant length of the capillary wave, which will be able to develop to large amplitudes in

a linear approximation. And then, at the resonant wavelength, its development is calculated using the numerical method described in [23].

In addition, an analytical or numerically analytical consideration of the non-linear development of surface waves and their interaction with shock flows (which are, in particular, pressure pulses of shock waves generated during the collapse of cavitation bubbles) was carried out in the works [22, 24–29] (Lavrentiev Institute of Hydrodynamics, Novosibirsk, Taganrog Institute of Technology of the Southern Federal University; University of Vienna, Vienna, Austria; American Mathematical Society). In these works, models of the nonlinear development of surface waves (the amplitude of the waves is comparable to their length) and their interaction with shock flows based on the integral laws of conservation of mass, momentum and Hugonio conditions were proposed. However, these models are applicable only for the case of shallow water, when the wavelength is large compared to the thickness of the liquid film. While the gas bubble is located in a volume whose size is much larger than the radius of the bubble, and its wall has an initial curvature. Therefore, the shallow water theory is obviously not applicable to describe waves on the surface of a gas bubble.

The publications [30–32] considered the propagation of shock waves in a liquid containing an ensemble of bubbles. The propagation of shock waves in a bubble environment was mainly considered by domestic authors. The presented publications took into account the influence of bubbles on the acoustic properties of the medium. However, collapse and coalescence were not considered (although, in particular, the mechanism of coalescence due to the Bjerknnes forces is known, the equations of bubble convergence due to the Bjerknnes force are known (presented by a member of the American Acoustic Society, the American Physical Society I.Sh. Akhatov, Moscow)) and crushing bubbles.

All this indicates the absence of consistent theoretical models describing the main mechanisms of increasing the interfacial surface under the influence of ultrasonic vibrations. The theoretical model proposed by the author is described in the following sections.

2. Problem statement

Within the framework of the task, vibrations of the interfacial surface are excited due to a spatially distributed force caused by shock waves under the action of collapsing cavitation bubbles. Since shock waves formed under the action of cavitation have a wide spectrum, the excitation of vibrations of the interfacial surface was initially considered when exposed to one of the harmonics having a fixed spatial wavelength and a fixed frequency.

In below, the hydrodynamics equations are presented in "usual" functions.

$$\operatorname{div}(\mathbf{u}) = 0, \quad (1)$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla p - \nabla f - \eta \Delta \mathbf{u} = 0, \quad (2)$$

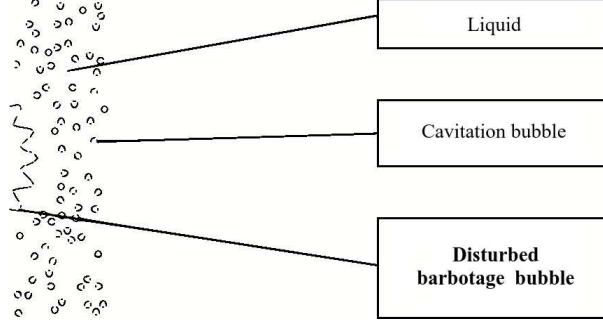


FIG. 1. Schematic representation of a perturbed small area of the bubble surface under the action of cavitation.

$$\frac{\partial h}{\partial t} = u_z, \quad (3)$$

$$-p + 2\eta \frac{\partial u_z}{\partial z} + \sigma \Delta h = 0, \quad (4)$$

$$\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0, \quad (5)$$

$$\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0, \quad (6)$$

where \mathbf{u} is vector of instantaneous velocity of liquid, m/s, ρ is liquid density, kg/m³, t is time, s, p is instantaneous pressure value in liquid, Pa, f is the potential of the volumetric force caused by the collapse of the cavitation bubble ensemble, Pa, η is dynamic viscosity of the liquid, Pa·s, h is instantaneous displacement profile of the interfacial surface, m, u_z is the component of the instantaneous velocity of the fluid along the z -axis, m/s, z is z -coordinate, m, σ is liquid surface tension, N/m, u_x is the component of the instantaneous velocity of the fluid along the x -axis, m/s, x is x -coordinate, m, u_y is the component of the instantaneous velocity of the fluid along the y -axis, m/s, y is y -coordinate, m.

Further, the equations will be presented as generalized functions equations. The functions are generalized and of slow growth, because the shock wave formed by cavitation bubble is scattered and attenuated in space and time due to liquid viscosity. The fact is followed from energy conservation law. For the slow growth generalized functions the convenient method of analysis is harmonic. Thus, a harmonic analysis of these equations was carried out, which is described in the next section.

3. Harmonic analysis of equations

In harmonic analysis, the potential of the volumetric force f was represented as an integral of harmonics (7). At the same time, only short-wave harmonics were considered, which provide vibrations of the surface with a wavelength much smaller than the radius of the bubbling bubble (8):

$$f(x, y, z, t) = \iiint_{\Omega_{k, \omega}} F(k_x, k_y, k_z, \omega_s, x, y, z) e^{I\omega_s t} dk_x dk_y dk_z d\omega_s, \quad (7)$$

$$\Omega_{k,\omega} = \mathbb{R}^4 \cap \overline{\{(k_x, k_y, k_z, \omega_s) \in \mathbb{R}^4 | (k_x^2 + k_y^2) < k_{\min}^2\}}, \quad (8)$$

$$F(k_x, k_y, k_z, \omega_s, x, y, z) = F_{k,\omega}(k_x, k_y, k_z, \omega_s) e^{Ik_x x + Ik_y y + Ik_z z}, \quad (9)$$

where F is the complex amplitude of the potential of the volumetric force caused by the collapse of the cavitation bubble, Pa, k_x is the component of the wavenumber along the x axis, m^{-1} , k_y is the component of the wavenumber along the y axis, m^{-1} , k_z is the component of the wavenumber along the z axis, m^{-1} , ω_s is round frequency, s^{-1} , I is imaginary unit, $\Omega_{k,\omega}$ is the range of harmonic parameter values, k_{\min} is the minimum value of the wavenumber, which limits the range of surface wave lengths from above in the framework of the problem statement, m^{-1} , \mathbb{R} is the set of real numbers, $F_{k,\omega}$ is the complex amplitude of a separate spatial harmonic of the volumetric force potential caused by the collapse of a cavitation bubble, Pa.

Using the specified representation, the equations of fluid motion (10), (11) with boundary conditions (12)–(15) look as follows:

$$\text{div}(\mathbf{U}) = 0, \quad (10)$$

$$\rho I \omega_s \mathbf{U} + \nabla P - \nabla F - \eta \Delta \mathbf{U} = 0, \quad (11)$$

$$I \omega_s H = U_z, \quad (12)$$

$$-P + 2\eta \frac{\partial U_z}{\partial z} + \sigma \Delta H = 0, \quad (13)$$

$$\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} = 0, \quad (14)$$

$$\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y} = 0, \quad (15)$$

where \mathbf{U} is complex amplitude (vector) of fluid velocity, m/s , ρ is liquid density, kg/m^3 , P is complex amplitude of liquid pressure, Pa, η is dynamic viscosity of the liquid, Pa·s, H is the complex amplitude of the displacement profile of the interfacial surface, m, U_z is the complex amplitude of the component of the velocity of the fluid along the z axis, m/s , σ is liquid surface tension, N/m, U_x is the complex amplitude of the component of the velocity of the fluid along the x axis, m/s , U_y is the complex amplitude of the component of the velocity of the fluid along the y axis, m/s .

To eliminate the volumetric force in the momentum conservation equation, the $P - F \rightarrow P$ was replaced. And as a result, the volumetric force "passed" into boundary conditions (19):

$$\text{div}(\mathbf{U}) = 0, \quad (16)$$

$$\rho I \omega_s \mathbf{U} + \nabla P - \eta \Delta \mathbf{U} = 0, \quad (17)$$

$$I \omega_s H = U_z, \quad (18)$$

$$-P + 2\eta \frac{\partial U_z}{\partial z} + \sigma \Delta H = F, \quad (19)$$

$$\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} = 0, \quad (20)$$

$$\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y} = 0, \quad (21)$$

where ρ is liquid density, kg/m^3 , η is dynamic viscosity of the liquid, $\text{Pa}\cdot\text{s}$, σ is liquid surface tension, N/m .

If additional conditions are imposed on the functions P, \mathbf{U} (22), (24), (25), (27) then the solution of this system of equations with boundary conditions exists and is unique:

$$\omega_s \neq 0, \quad (22)$$

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}, \quad (23)$$

$$\forall i = 1, \dots, 4 \quad \left(\begin{pmatrix} P \\ U_x \\ U_y \\ U_z \end{pmatrix}, \mathbf{e}_i \right) \in C^\infty(S), \quad (24)$$

$$P(x + nX, y + mY, z) = P(x, y, z), \quad (25)$$

$$\lim_{z \rightarrow \infty} \max_{(x, y) \in \mathbb{R}^2} |P| = 0, \quad (26)$$

$$\mathbf{U}(x + nX, y + mY, z) = \mathbf{U}(x, y, z), \quad (27)$$

$$\lim_{z \rightarrow \infty} \max_{(x, y) \in \mathbb{R}^2} |\mathbf{U}| = 0, \quad (28)$$

$$n, m \in \mathbb{Z}, \quad (29)$$

where S is function definition area, \mathbf{e}_i is unit vector in \mathbb{R}^4 , \mathbb{Z} is the set of integers.

An additional condition for the harmonic wavenumber is due to the fact that short waves are considered, which have a decisive effect on the formation of the interfacial surface according to experimental data from [33] and which are small in length compared with the radius of curvature of the undisturbed surface. To prove the existence of an exact solution, a method of complex amplitudes was used:

$$\mathbf{U} = \mathbf{V} + \nabla\Phi, \quad (30)$$

$$\begin{pmatrix} H \\ U_x \\ U_y \\ U_z \end{pmatrix} = \begin{pmatrix} H_A e^{Ik_x x + Ik_y y} \\ V_{A,x} e^{Ik_x x + Ik_y y - \alpha z} \\ V_{A,y} e^{Ik_x x + Ik_y y - \alpha z} \\ V_{A,z} e^{Ik_x x + Ik_y y - \alpha z} \end{pmatrix} + \begin{pmatrix} 0 \\ Ik_x \Phi_A e^{Ik_x x + Ik_y y - kz} \\ Ik_y \Phi_A e^{Ik_x x + Ik_y y - kz} \\ -k \Phi_A e^{Ik_x x + Ik_y y - kz} \end{pmatrix}, \quad (31)$$

where \mathbf{V} is the complex amplitude (vector) of the vortex component of the fluid velocity, m/s , Φ is the complex amplitude of the potential of the potential component of the fluid velocity, m^2/s , H_A is the complex amplitude of the spatial harmonic of the displacement profile of the interfacial surface, m , $V_{A,x}$ is the complex amplitude of the spatial harmonic component of the vortex component of the velocity of the fluid along the x axis, m/s , α is a complex coefficient characterizing the decrease in the amplitude of the vortex component of velocity along the z axis, m^{-1} , $V_{A,y}$ is the complex amplitude of the spatial harmonic component of the vortex component of the velocity of the fluid along the y axis, m/s , $V_{A,z}$ is the complex amplitude of the spatial harmonic component of the vortex component of the velocity of the fluid

along the z axis, m/s , Φ_A is the complex amplitude of the spatial harmonic potential of the potential component of the fluid velocity, m^2/s , k is the modulus of projection of the wavenumber onto the plane Oxy , m^{-1} .

According to the method of complex amplitudes, the problem is reduced to solving the following system of linear algebraic equations:

$$\begin{pmatrix} 0 & 0 & Ik_x & Ik_y & -\alpha \\ -\sigma k^2 & 2\eta k^2 + I\omega_s \rho & 0 & 0 & -2\eta\alpha \\ 0 & -2Ik_x k & -\alpha & 0 & Ik_x \\ 0 & -2Ik_y k & 0 & -\alpha & Ik_y \\ I\omega_s & k & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} H_A \\ \Phi_A \\ V_{A,x} \\ V_{A,y} \\ V_{A,z} \end{pmatrix} = \begin{pmatrix} 0 \\ F_{k,\omega} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (32)$$

The main idea of prove is based on complex polynomes algebra. The existence follows from

$$\det \begin{pmatrix} 0 & 0 & Ik_x & Ik_y & -\alpha \\ -\sigma k^2 & 2\eta k^2 + I\omega_s \rho & 0 & 0 & -2\eta\alpha \\ 0 & -2Ik_x k & -\alpha & 0 & Ik_x \\ 0 & -2Ik_y k & 0 & -\alpha & Ik_y \\ I\omega_s & k & 0 & 0 & -1 \end{pmatrix} \neq 0.$$

The generalized functions problem statement with provings based on the harmonic analysis was presented in following section.

4. Generalized solution of equations

When setting the problem in terms of generalized functions (of slow growth), it is assumed that the disturbance of the interfacial surface is influenced by collapsing cavitation bubbles that occur randomly in different places during each period of ultrasonic vibrations. It is assumed that no more than one collapsing cavitation bubble occurs in each oscillation period in the considered computational domain. In this case, the bubbles can be located at distances from the set of $[x_{\min}; x_{\max}] \times [y_{\min}; y_{\max}] \times [z_{\min}; z_{\max}]$ (x_{\min} is the minimum coordinate of the occurrence of a collapsing cavitation bubble in x , m , x_{\max} is the maximum coordinate of the occurrence of a collapsing cavitation bubble in x , m , y_{\min} is the minimum coordinate of the occurrence of a collapsing cavitation bubble in y , m , y_{\max} is the maximum coordinate of the occurrence of a collapsing cavitation bubble in y , m , z_{\min} is the minimum coordinate of the occurrence of a collapsing cavitation bubble in z , m , z_{\max} is the maximum coordinate of the occurrence of a collapsing cavitation bubble in z , m) ($z_{\min} > 0$; $z_{\max} > z_{\min}$; $y_{\max} > y_{\min}$; $x_{\max} > x_{\min}$).

Next, the designations are introduced:

$$\begin{aligned} \Omega_{bub} &= [0; 1], \quad M : \mathbb{Z} \rightarrow \Omega_{bub}, \\ \mathbf{r}_{bub} : \Omega_{bub} &\rightarrow ([x_{\min}; x_{\max}] \times [y_{\min}; y_{\max}] \times [z_{\min}; z_{\max}]) \\ \mathbf{r}_{bub} &= \begin{pmatrix} x_{bub} \\ y_{bub} \\ z_{bub} \end{pmatrix}, \quad \varphi_n \in \mathbb{S}(\mathbb{R}^n), \end{aligned}$$

$$\begin{aligned}
L_{4,3} : \mathbb{S}(\mathbb{R}^4) &\rightarrow \mathbb{S}(\mathbb{R}^3), & L_{4,3}[\varphi_4(t, x, y, z)] &= \varphi_3(t, x, y) \\
\varphi_3(t, x, y) &= \varphi_4(t, x, y, 0), & C_4 &= \mathbb{S}(\mathbb{R}^4), & C_3 &= \mathbb{S}(\mathbb{R}^3), \\
\vec{\varphi}_4 &= \begin{pmatrix} \varphi_{4,1} \\ \varphi_{4,2} \\ \varphi_{4,3} \end{pmatrix}, & V_{bub} &= (x_{\max} - x_{\min})(y_{\max} - y_{\min})(z_{\max} - z_{in}), & P_{bub} &= n_{bub}V_{bub},
\end{aligned}$$

where Ω_{bub} is a continuous set of elementary outcomes in the collapse of a cavitation bubble, M is a function characterizing the random process of collapse of cavitation bubbles, \mathbf{r}_{bub} is a function that determines the positions of cavitation bubbles, m^{-3} , φ_n is smooth function with rapidly decreasing of space \mathbb{R}^n , \mathbb{S} is the Schwartz space of basic rapidly decreasing functions, $L_{4,3}$ is projector of smooth function with rapidly decreasing of \mathbb{R}^4 to smooth function with rapidly decreasing of \mathbb{R}^3 , C_4 is the space of basic functions in the volume of a liquid, C_3 is the space of basic functions on the surface of a liquid, $\vec{\varphi}_4$ is three-dimensional vector of basic functions from 4 arguments (t, x, y, z) , V_{bub} is the volume of the area of possible appearance of a cavitation bubble, m^3 , P_{bub} is the probability of a collapsing cavitation bubble in 1 period of oscillation, n_{bub} is concentration of cavitation bubbles, m^{-3} .

In these notations $[x_{\min}; x_{\max}]$ (and also for y -, z -coordinates) square brackets denote the close segment including boundary values.

The notation $L_{4,3}[\varphi_4(t, x, y, z)]$ denotes action of projection operation $L_{4,3}$ on function φ_4 .

Using these notations, the problem in terms of generalized functions is formulated as follows:

$$(\text{div}(\mathbf{u}), q) = \left(\frac{\partial h}{\partial t}, L_{4,3}[q] \right) \quad (33)$$

$$\left(\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla p - \eta \Delta \mathbf{u}, \vec{\varphi}_4 \right) = (f - \sigma \Delta h, L_{4,3}[\varphi_{4,3}]), \quad (34)$$

where q is smooth function with rapidly decreasing of space \mathbb{R}^4 .

The surface profile h is a linear continuous functional over the 3-dimensional space of basic functions C_3 , and the components of the velocity vector u_i and pressure p are linear continuous functionals over the 4-dimensional space of basic functions C_4 . If the function f is harmonic in coordinates and time, then the solution of these equations (33), (34) are regular generalized functions. For the case of bubble collapse, the function f is represented as follows:

$$\begin{aligned}
(\mathbf{s}, \mathbf{s}_{bub}(i), \mathbf{k}_2) &= \left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x_{bub}(L_1(\frac{M(i)}) \\ y_{bub}(L_1(\frac{M(i)}) \end{pmatrix}), \begin{pmatrix} k_x \\ k_y \end{pmatrix} \right), \\
G(i, x, y) &= \iiint_{k_z \in \mathbb{R}, k_x^2 + k_y^2 \geq k_{\min}^2} \frac{e^{I((\mathbf{k}_2, \mathbf{s} - \mathbf{s}_{bub}(i)) - k_z z_{bub}(L_1(\frac{M(i)}{P_{bub}})))}}{8\pi^3 k_{total}^2} dk_x dk_y dk_z, \\
\varphi_{3,3} &= L_{4,3}[\varphi_{4,3}], \\
(f, \varphi_{3,3}) &= - \sum_{i=-\infty}^{\infty} \iint_{\mathbb{R}^2} W_{bub} J_1\left(\frac{M(i)}{P_{bub}}\right) G(i, x, y) \varphi_{3,3}(iT, x, y) dx dy, \quad (35)
\end{aligned}$$

where L_1 is a function equal to 1 if the argument is greater than 1 and the argument itself otherwise, k_{total} is the modulus of the wavenumber, m^{-1} , W_{bub} is the proportionality coefficient determining the energy of the shock wave during the collapse of one cavitation bubble, N/m^4 , J_1 is indicator function equal to 1 if the argument is less than 1 and 0 otherwise, T is the period of ultrasonic vibrations, s.

It is further shown that the function $G(i, x, y)$ is infinitely differentiable in x, y after performing the following transformations of the integral

$$\iiint_{k_z \in \mathbb{R}, k_x^2 + k_y^2 \geq k_{\min}^2} \frac{e^{I((\mathbf{k}_2, \mathbf{s} - \mathbf{s}_{bub}(i)) - k_z z_{bub}(L_1(\frac{M(i)}{P_{bub}})))}}{8\pi^3 k_{total}^2} dk_x dk_y dk_z$$

using residue theory and Jordan's lemma.

$$\begin{aligned} & \iiint_{k_z \in \mathbb{R}, k_x^2 + k_y^2 \geq k_{\min}^2} \frac{e^{I((\mathbf{k}_2, \mathbf{s} - \mathbf{s}_{bub}(i)) - k_z z_{bub}(L_1(\frac{M(i)}{P_{bub}})))}}{8\pi^3 k_{total}^2} dk_x dk_y dk_z \\ &= \frac{1}{8\pi^3} \iint_{\|\mathbf{k}_2\|^2 \geq k_{\min}^2} e^{I(\mathbf{k}_2, \mathbf{s} - \mathbf{s}_{bub}(i))} \int_{k_z \in \mathbb{R}} \frac{e^{-Ik_z z_{bub}(L_1(\frac{M(i)}{P_{bub}}))}}{\|\mathbf{k}_2\|^2 + k_z^2} dk_z dk_x dk_y \\ &= \frac{1}{8\pi^2} \iint_{\|\mathbf{k}_2\|^2 \geq k_{\min}^2} e^{I(\mathbf{k}_2, \mathbf{s} - \mathbf{s}_{bub}(i))} \frac{e^{-\|\mathbf{k}_2\| z_{bub}(L_1(\frac{M(i)}{P_{bub}}))}}{\|\mathbf{k}_2\|} dk_x dk_y \quad (36) \end{aligned}$$

Further, the infinite differentiability of the function $G(i, x, y)$ follows from the uniform convergence of the integral

$$\iint_{\|\mathbf{k}_2\|^2 \geq k_{\min}^2} e^{I(\mathbf{k}_2, \mathbf{s} - \mathbf{s}_{bub}(i))} \frac{e^{-\|\mathbf{k}_2\| z_{bub}(L_1(\frac{M(i)}{P_{bub}}))}}{\|\mathbf{k}_2\|} dk_x dk_y$$

when taking the partial derivative of any integer order with any multiindex from the integrand. Uniform convergence is due to the fact that any partial derivative of the integrand is a polynomial of \mathbf{k}_2 multiplied by a complex function whose modulus decreases exponentially with the norm \mathbf{k}_2 tending to plus infinity.

The physical meaning of the proposed Fourier representation of the volumetric force is that each bubble is a source of force, the Laplacian of which is a delta function of 4 arguments (3 coordinates and time), with the exception of long-wave harmonics, the wavenumber of the projection of which on the surface of the liquid gas is less than k_{\min} .

Further, the proof of the fact is carried out that the capillary wave profile function during the collapse of a single bubble generating a volumetric force with a Laplacian in the form of a Dirac delta function will represent a regular generalized function.

4.1. Proof of regularity of h-generalized function (surface profile) at single bubble collapsing. When a single cavitation bubble collapses at time 0,

the generalized force function is represented as follows:

$$(f_{bub}, \varphi_{3,3}) = \iint_{\mathbb{R}^2} W_{bub} G(0, x, y) \varphi_{3,3}(0, x, y) dx dy, \quad (37)$$

where f_{bub} is the potential of the volumetric force caused by the collapse of the single cavitation bubble, Pa.

To prove the regularity of the generalized profile function, it is sufficient to prove the convergence of the improper integral:

$$\iiint_{\omega_s, k_z \in \mathbb{R}; k_x^2 + k_y^2 \geq k_{\min}^2} \frac{e^{I((\mathbf{k}_2, \mathbf{s} - \mathbf{s}_{bub}) - k_z z_{bub})}}{8\pi^3 k_{total}^2} \frac{\det B_k}{\det A_k} \frac{e^{I\omega_s t}}{2\pi} d\omega_s dk_x dk_y dk_z, \quad (38)$$

$$A_k = \begin{pmatrix} 0 & 0 & Ik_x & Ik_y & -\alpha \\ -\sigma k^2 & 2\eta k^2 + I\omega_s \rho & 0 & 0 & -2\eta\alpha \\ 0 & -2Ik_x k & -\alpha & 0 & Ik_x \\ 0 & -2Ik_y k & 0 & -\alpha & Ik_y \\ I\omega_s & k & 0 & 0 & -1 \end{pmatrix}, \quad (39)$$

$$B_k = \begin{pmatrix} 0 & Ik_x & Ik_y & -\alpha \\ 2Ik_x k & -\alpha & 0 & Ik_x \\ 2Ik_y k & 0 & -\alpha & Ik_y \\ -k & 0 & 0 & -1 \end{pmatrix}. \quad (40)$$

The physical meaning of the integral (38) with subsequent multiplication of W_{bub} consists in the "usual" function of the profile of the interfacial surface h from the coordinates x, y and time t .

The base stages of proof of convergence of integral (38) are following.

1. The transformation of an integral to an integral of a polynomial function:

$$\iiint_{\omega_s, k_z \in \mathbb{R}; k_x^2 + k_y^2 \geq k_{\min}^2} \frac{e^{I((\mathbf{k}_2, \mathbf{s} - \mathbf{s}_{bub}) - k_z z_{bub} + \omega_s t)}}{16\pi^4 k_{total}^2} \frac{\det B_k}{\det A_k} d\omega_s dk_x dk_y dk_z. \quad (41)$$

If $\omega_s = 0$, then

$$\frac{\det B_k}{\det A_k} = 0. \quad (42)$$

If $\omega_s \neq 0$, then

$$\frac{\det B_k}{\det A_k} = \frac{k}{\sigma k^3 + 4\eta k^2 I\omega_s - \omega_s^2 \rho + \frac{4\eta^2 k^4}{\rho} \left(1 - \left(1 + \frac{I\omega_s \rho}{\eta k^2}\right)^{\frac{1}{2}}\right)}. \quad (43)$$

2. Proof of convergence of integral by frequency only. The transformations of the integral given in below:

$$v.p. \int_{\omega_s \in \mathbb{R}} \frac{e^{I\omega_s t}}{\sigma k^3 + 4\eta k^2 I\omega_s - \omega_s^2 \rho + \frac{4\eta^2 k^4}{\rho} \left(1 - \left(1 + \frac{I\omega_s \rho}{\eta k^2}\right)^{\frac{1}{2}}\right)} d\omega_s, \quad (44)$$

$$\Omega_{s,k} = \frac{\rho \omega_s}{\eta k^2},$$

$$v.p. \int_{\Omega_{s,k} \in \mathbb{R}} \frac{e^{I\omega_s t}}{\eta k^2 \left(\frac{\rho\sigma}{\eta^2 k} + 4I\Omega_{s,k} - \Omega_{s,k}^2 + 4(1 - (1 + I\Omega_{s,k})^{\frac{1}{2}}) \right)} d\Omega_{s,k} \quad (45)$$

where $\Omega_{s,k}$ is dimensionless circular frequency.

Sign of square root in expressions $(1 + \frac{I\omega_s \rho}{\eta k^2})^{\frac{1}{2}}$, $(1 + I\Omega_{s,k})^{\frac{1}{2}}$ is choosed for positive real part due to attenuation of liquid oscillations with z -increasing.

The convergence of integral is followed from:

- (a) According to the complex algebra of polynomes (the mathematical derivations of this are omitted from the paper), the absolute value of the denominator in subintegral function (45) is more or equals positive constant D ($D > 0$). The denominator may be zero, if ω_s contains positive imaginary part (the attenuation of waves during time due to liquid viscosity).
- (b) The asymptotics of the denominator at large ω_s and fixed k is $O(\omega_s^2)$.

According to the complex algebra of polynomes, large k the asymptotics of the integral (45) is $O(\frac{1}{k})$.

3. To notice, that

$$\int_{k_z \in \mathbb{R}} \frac{e^{-Ik_z z_{bub}}}{k_{total}^2} dk_z = \frac{\pi e^{-kz_{bub}}}{k}. \quad (46)$$

4. In finish of the proof to notice, due to decreasing exponent in (46), the integral (38) is reduced to absolutely converged integral by k_x and k_y only, which contains in subintegral production:

- (a) Complex exponent $e^{I(k_x x + k_y y)}$.
- (b) Descreasing exponent $e^{-kz_{bub}}$.
- (c) Function, which is determined at $\forall k \geq k_{min}$ and limited (at large k) by fractional-polynomial function of k .

According to the complex of conditions the integral (38) is not only absolutely converged, but also infinitely differentiable in x, y .

The integral (38) is less than $\frac{D_1}{1+(x^2+y^2)^{\frac{1}{2}}}$, $D_1 > 0$. The fact is proofed by following steps.

1. Representation of complex exponent (at large $x^2 + y^2$) as

$$e^{I(k_x x + k_y y)} = \Delta_k \left(-\frac{e^{I(k_x x + k_y y)}}{x^2 + y^2} \right) \quad \left(\Delta_k = \frac{\partial^2}{\partial k_x^2} + \frac{\partial^2}{\partial k_y^2} \right).$$

2. Representation of the subintegral expression as

$$\begin{aligned} e^{I(k_x x + k_y y) - kz_{bub}} F_1(k) &= \Delta_k \left(-\frac{e^{I(k_x x + k_y y)}}{x^2 + y^2} \right) e^{-kz_{bub}} F_1(k) \\ &= -\operatorname{div} \left(\nabla_k \left(\frac{e^{I(k_x x + k_y y)}}{x^2 + y^2} \right) e^{-kz_{bub}} F_1(k) \right) \\ &\quad + \left(\nabla_k \left(\frac{e^{I(k_x x + k_y y)}}{x^2 + y^2} \right), \nabla_k (e^{-kz_{bub}} F_1(k)) \right) \end{aligned}$$

(F_1 is function on k with fractional-polynomial asymptotics at large k).

3. According to the Gauss and Ostrogradsky theorem the integral of divergence in previous expression is zero. Thus, the estimation was proofed.

The constant D_1 is general for all times. Thus, the integral

$$\iiint h(t, x, y) \phi_3(t, x, y) dt dx dy$$

for any smooth function ϕ_3 with rapidly decreasing is converged and continuous functional from ϕ_3 . In finish, the profile of surface due to "delta function" bubble collapse is regular generalized function of slow growth.

To proof of regularity of the profile of surface for multiple bubbles, it is necessary to estimate of time asymptotics of the integral (45).

The time asymptotics were estimated by residue theory and Jordan's lemma. At $k \geq k_{\min} > 0$ the subintegral expression (45) has poles of 1st order. The poles was determined by solving of quartic polynomial equation (48) using Ferrari's method:

$$\beta_k(k) = \frac{\rho\sigma}{\eta^2 k} \quad (47)$$

$$\Omega_{s,k}^4 - 8I\Omega_{s,k}^3 - 2(\beta_k + 12)\Omega_{s,k}^2 + 8I(\beta_k + 2)\Omega_{s,k} + \beta_k^2 + 8\beta_k = 0. \quad (48)$$

The roots of denominator of

$$\eta k^2 \left(\frac{\rho\sigma}{\eta^2 k} + 4I\Omega_{s,k} - \Omega_{s,k}^2 + 4(1 - (1 + I\Omega_{s,k})^{\frac{1}{2}}) \right)$$

are subset of roots of quartic polynomial equation (48). At large k (or small β_k) the asymptotics of root is $\frac{1}{2}\beta_k + o(\beta_k)$. Thus, at large k the integral (45) equals to zero at $t < 0$ and has asymptotics $P_k(k)e^{-\frac{\beta_k \eta k^2}{2\rho} t}$ ($P_k(k)$ is fractional and polynomial function on k). The small β_k asymptotics was calculated by series decomposition of analytical solutions obtained by Ferrari's method. For any k , the asymptotics of the integral is less than $|Q_k(k)|e^{-A_k|t|}$ ($Q_k(k)$ is fractional and polynomial function on k , $A_k > A_0(k_{begin}, k_{end}) > 0$ for k in finite range $[k_{begin}, k_{end}]$; $k_{begin} \geq k_{\min} > 0$). The inequality $A_k > A_0 > 0$ was proofed using the following facts:

- 0 isn't root of (48) $\forall k \in [k_{begin}, k_{end}]$;
- left part of (48) is continuous by $\Omega_{s,k}$;
- $[k_{begin}, k_{end}]$ is compact set.

The exponential decreasing during time follows the convergence of series $\sum_{i=-\infty}^{i=\infty} |G(i, x, y)|$. Thus, the profile of surface is regular generalized function of slow growth. Further the estimated dependences of interfacial surface area (on ultrasonic influence modes and liquid properties) was obtained.

5. Estimated dependences of interfacial surface

The relative increase in the interfacial surface was estimated according to the following expression (49) (the value of h is represented as a "ordinary" function

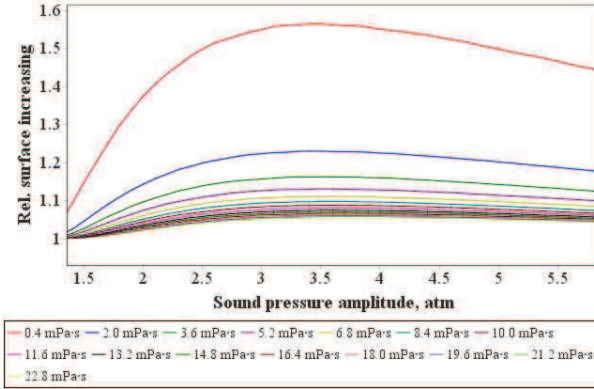


Fig. 2. Estimated dependence of the average increase in the interfacial surface on the amplitude of the sound pressure.

according to the previously presented proof). Integration in the expression (49) is understood in the sense of the main Cauchy value. Also the dependences take into account inhomogeneous acoustic field near interfacial surface:

$$K_{rel} = \frac{\iiint_{(s \in [x_{min}; x_{max}] \times [y_{min}; y_{max}], (t \in [0; T]))} (1 + \|\nabla h\|^2)^{\frac{1}{2}} dx dy dt}{T(x_{max} - x_{min})(y_{max} - y_{min})}. \quad (49)$$

where K_{rel} is relative increase in the interfacial surface.

Further, on the basis of the law of conservation of energy (the elementary act of collapse of a cavitation bubble leads to the fact that the liquid acquires additional kinetic energy), the estimated dependences of the increase in the interfacial surface were revealed. The dependences are given in the next figures.

The dependences were revealed for a liquid with a density of 1000 kg/m^{-3} and a surface tension of 0.072 N/m . The frequency of ultrasonic vibrations was 22 kHz . When constructing the dependencies, the dependence of the concentration of cavitation bubbles on the amplitude of the sound pressure was taken into account. A phenomenological model of the formation of a cavitation region was used to determine the concentration of cavitation bubbles, given in [34]. When estimating the energy passing into an increase in the interfacial surface, the coefficient W_{bub} was additionally multiplied by a correction factor taking into account the transition of part of the energy during the collapse of cavitation bubbles into heat. Figure 2 shows the estimated dependence of the average increase in the interfacial surface on the amplitude of the sound pressure.

The presented dependences indicate that the increase in the interfacial surface reaches 1.6 times, which is close to the experimental data obtained by the method of high-speed filming [33].

The figure 3 shows estimated dependence of the maximum increase in the interfacial surface on viscosity. The maximum was founded on fixed viscosity (and another liquid properties) and ultrasonic oscillations frequency over sound pressure amplitude.

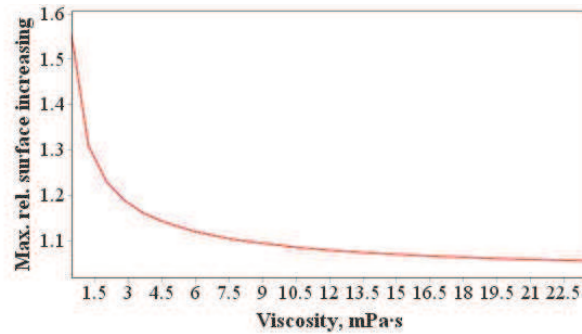


Fig. 3. Estimated dependence of the maximum increase in the interfacial surface on viscosity.

The obtained dependences of the interfacial surface on viscosity indicate the existence of a limiting viscosity, starting from which the effect stops. The dependence in the figure 3 was shown, that the interfacial surface is rapidly decreased with viscosity growth. However the cavitation at the presented viscosities range remains advanced [34]. This indicates the need for theoretical and experimental research at different ambient temperatures. Since, on the one hand, viscosity decreases with increasing temperature, and on the other hand, the degree of cavitation development decreases. Apparently, there may be an optimal temperature in this regard.

6. Conclusion

The model of the formation of linear short capillary waves on the liquid-gas surface under the action of cavitation created by ultrasonic vibrations was proposed. The equations of propagation of capillary waves were constructed in the formulation of classical and generalized functions (of slow growth) that take into account: the viscosity of the liquid phase; attenuation of wave vibrations over time due to the viscosity of the liquid phase, which implies a limited amplitude of the waves (despite the fact that in the absence of attenuation, the wave can oscillate indefinitely over time). It was proved, that for equations in generalized functions for the case of collapse of a set of bubbles in a limited volume of liquid, the displacement profile (as generalized function, with is integral in the sense of the principal Cauchy value) of the interfacial surface is a regular generalized function of slow growth. The estimated dependences of the average increase in the interfacial surface on the parameters of ultrasonic action and the viscosity of the liquid are constructed. The dependences showed an increase in the interfacial surface up to 1.6 times or more for a liquid with a viscosity close to water. The obtained value is similar to the experimental data. The existence of a limiting viscosity has been established, starting from which the effect ceases to be noticeable. This indicates the need for research at different ambient temperatures. Since, on the one hand, with increasing temperature, the viscosity of the liquid phase decreases, and on the other hand, the degree of cavitation development decreases. Apparently, there may be an optimal temperature in this regard.

REFERENCES

1. Novoselov A. G., Dujij A. B., and Golikova E. Yu., “Molecular diffusion of gases in a liquid. Coefficients of molecular diffusion of carbon dioxide in water [in Russian],” *Sci. J. Processes Food Prod. Equip.*, No. 2 (2014).
2. Podryga V. O., Vikhrov E. V., and Polyakov S. V., “Molecular dynamic calculation of the gas diffusion coefficient on the example of argon, nitrogen, hydrogen, oxygen, methane and carbon dioxide [in Russian],” *Prepr. Keldysh Inst. Appl. Math.*, No. 96 (2019). doi:10.20948/prepr-2019-96
3. Shadrin E. Y., Anufriev I. S., and Sharypov O. V., “Investigation of the process of spraying and burning coal-water fuel using a pneumatic nozzle [in Russian],” *Appl. Mech. Tech. Phys.*, **62**, No. 3, 165–171 (2021).
4. Khmelev V. N., Shalunov A. V., Golykh R. N., Nesterov V. A., Dorovskikh R. S., and Shalunova A. V., “Determination of the modes and the conditions of ultrasonic spraying providing specified productivity and dispersed characteristics of the aerosol,” *J. Appl. Fluid Mech.*, **10**, No. 5, 1409–1419 (2017).
5. Rozenberg L. D., *Physical Foundations of Ultrasonic Technologies [in Russian]*, Nauka, Moscow (1970).
6. Golykh R. N., “Evaluation of optimum modes and conditions of cavitation and acoustic absorption intensification for increasing,” *J. Appl. Fluid Mech.*, **10**, No. 5, 1235–1246 (2017).
7. Rozenberg L. D., *Powerful Ultrasonic Fields [in Russian]*, Nauka, Moscow (1968).
8. Morton J., Khavari M., Priyadarshi A., Kaur A., Grobert N., Mi J., Porfyakis K., Prentice P., Eskin D., and Tzanakis I., “Dual frequency ultrasonic cavitation in various liquids: High-speed imaging and acoustic pressure measurements,” *Phys. Fluids*, **35** (2023).
9. Maltsev N. N., *Absorption of Benzene and the Possibility of its Intensification by Ultrasound [in Russian]*, Dnepropetr. Chem. Technol. Inst., Dnepropetrovsk (1956).
10. Tan M., Friend J., Matar O., and Yeo L., “Capillary wave motion excited by high frequency surface acoustic waves,” *Phys. Fluids*, No. 22, 112112 (2023). doi:10.1063/1.3505044
11. Wallenberger P. and Lyzenga D. R., “Measurement of the surface tension of water using microwave backscatter from gravity-capillary waves,” *IEEE Trans. Geosci. Remote Sensing*, **28**, No. 6, 1012–1016 (1990). 10.1109/36.62625.
12. Taller D. and Go D., “Modulated exponential films generated by surface acoustic waves and their role in liquid wicking and aerosolization at a pinned drop,” *Phys. Rev. E, Stat. Nonlinear Soft Matter Phys.*, No. 87, 53004 (2013).
13. Punzmann H., Shats M., and Xia H., “Phase randomization of three-wave interactions in capillary waves,” *Phys. Review Lett.*, No. 103, 26946 (2009). doi:10.1103/PhysRevLett.103.064502
14. Xu J. and Attinger D., “Acoustic excitation of superharmonic capillary waves on a meniscus in a planar micro-geometry,” *Phys. Fluids*, No. 19 (2009). doi:10.1063/1.2790968
15. Sugondo A., Sutrisno T., Anggono W., and Anne O., “Effect of frequency on droplet characteristics in ultrasonic atomization process,” *E3S Web Conf.*, **19**, 1002 (2019). doi:10.1051/e3sconf/201913001002
16. Bonn D. and Wegdam G., “Capillary waves and ellipsometry experiments,” *J. Phys. I, France*, **2**, N. 19, 1755–1764 (1992). doi:10.1051/jp1:1992242
17. Shen L., Denner F., Morgan N., Wachem B., and Dini D., “Capillary waves with surface viscosity,” *J. Fluid Mech.*, No. 847, 644–663 (2018). doi:10.1017/jfm.2018.364
18. Rahimzadeh A., Ahmadian Y.M.R., and Eslamian M., “Experimental study on the characteristics of capillary surface waves on a liquid film on an ultrasonically vibrated substrate,” *Fluid Dyn. Res.*, No. 50 (2018). doi:10.1088/1873-7005/aae446
19. Ehrhorn J. and Semke W., “Numerical prediction of vibration induced liquid atomization,” *Int. J. Nov. Res. Eng. Pharm. Sci.*, **1**, No. 3, 1–9 (2014).
20. Lugovskoy A. and Lyashok A., “Physical analogue of the process of ultrasonic liquid nebulisation in a thin layer,” *J. Mech. Eng. Kyiv Polytech. Inst.*, 110–114 (2013).
21. Simon J. C., Sapozhnikov O. A., Khokhlova V. A., Crum L. A., and Bailey M. R., “Ultrasonic atomization of liquids in drop-chain acoustic fountains,” *J. Fluid Mech.*, No. 766, 129–146 (2015).

22. *Ostapenko V. V.*, “On the laws of conservation of shallow water theory [in Russian],” *Dokl. Akad. Nauk*, **464**, No. 5, 558–561 (2015).
23. *Schmidmayer K., Petitpas F., Daniel E., Favrie N., and Gavriluk S.*, “A model and numerical method for compressible flows with capillary effects,” *J. Comput. Phys.*, No. 334, 468–496 (2017).
24. *Ostapenko V. V.*, “Modified equations of shallow water theory allowing for the propagation of discontinuous waves along a dry riverbed [in Russian],” *Appl. Mech. Tech. Phys.*, **48**, No. 6, 22–43 (2007).
25. *Abbasov I. B.*, “Numerical simulation of nonlinear surface gravity waves transformation under shallow-water conditions [in Russian],” *Appl. Math.*, No. 3, 135–141 (2012).
26. *Lannes D. and Marche F.*, “Nonlinear wave-current interactions in shallow water,” *Stud. Appl. Math.*, **136**, No. 4, 382–423 (2016).
27. *Constantin A.*, *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis*, SIAM, Philadelphia, PA (2011) (CBMS-NSF Reg. Conf. Ser. Appl. Math.; vol. 81).
28. *Düll W. P.*, “On the mathematical description of water waves,” arXiv:1612.06242 (2016).
29. *Lannes D.*, *The Water Waves Problem: Mathematical Analysis and Asymptotics*, Amer. Math. Soc., Providence, RI (2013).
30. *Ogorodnikov I.*, “Reflection of sound pulses from an inhomogeneous bubble medium,” *J. Phys., Conf. Ser.*, **2057**, 12032 (2021). doi:10.1088/1742-6596/2057/1/012032
31. *Ogorodnikov A.*, “The formation of nonlinear sound fields in the boundary region of the bubble medium,” *J. Phys., Conf. Ser.*, **1677**, 12144 (2020). doi:10.1088/1742-6596/1677/1/012144
32. *Gimaltdinov I., Gizzatullina A., and Gimaltdinova A.*, “On the issue of initiation of bubble detonation by small-amplitude waves,” *IOP Conf. Ser. Materials Sci. Eng.*, **919**, 62060 (2020). doi:10.1088/1742-6596/1677/1/012144
33. *Golykh R. N., Carrat J.-B., Khmelev V. N., Manyakhin I. A., Minakov V. D., Genne D. V., and Barsukov A. R.*, “Effect of ultrasonic cavitation on the gas-liquid interface under forced aeration,” *J. Appl. Mech. Tech. Phys.* **65**, No. 6, 1082–1095 (2024).
34. *Golykh R., Shalunov A., Khmelev V., Lopatin R., Minakov V., and Shakura V.*, “Evaluation of optimum modes and conditions providing increasing ultrasonic cavitation area in high-viscous and non-newtonian fluids,” *Rom. J. Acoustics Vibration*, **17**, No. 2, 101–108 (2020).

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